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fuzzy complex valued metric spaces

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Abstract:
This present research paper introduces Fuzzy Complex Valued Metric Spaces, which are novel extensions of classical metric spaces embedding both fuzziness and complex-valued distances in a unique way. FCVMS embeds all the uncertainty handling capabilities of fuzzy metric spaces along with multi-dimensional phase and magnitude relationships that are inherent in complex-valued metrics. Properties which form the topological structure of FCVMS include open and closed sets, interior, closure, boundary, behavior of sequences w.r.t. convergence, completeness, compactness, separability, and connectedness. As for a number of case studies to show the applicability of FCVMS, two have been carried out: the first relates to modeling uncertainty and oscillations in control systems, while the second concerns simulating sensor networks, where both imprecise measurement and periodic signal fluctuations form an essential issue.
Graphs and numerical simulations provide an idea of how FCVMS can model a system better than other conventional metrics. The results show that FCVMS is a strong framework for analyzing systems that are dynamic, uncertain, and phase-sensitive; they range from engineering and telecommunications to environmental monitoring.

Keywords:
(Fuzzy complex valued metric spaces, fuzzy metric spaces, complex-valued metrics, topological structure, uncertainty modeling, periodic behavior, control systems, sensor networks, numerical simulations, convergence, compactness)

Introduction

In classical analysis, metric spaces are a standard framework in which notions such as distance, continuity, and convergence are set. After some time, several generalizations of metric spaces appeared to host more involved and subtle mathematical structures**(Sutherland, 2009)**. One such generalization is the fuzzy metric space, put forth to account for situations where uncertainty or vagueness should play a critical role. In fuzzy metric spaces, the fuzziness concept developed upon a cornerstone given by the fuzzy set theory of Zadeh is introduced**(Grabisch et al., 2013)**. This permits more flexibility in the interpretation of the distance. Some applications have been found in decision theory, optimization, and control systems, where metrics may not work properly**(da Fonseca, 1995)**.

Complex metric spaces have more recently been developed to handle those spaces whose metric is complex-valued instead of real-valued. The complex metric space can model phenomena where magnitude as well as direction-or phase-play a role in the distance between points, hence have been of particular use in fields such as quantum physics and signal processing**(Jeffrey, 2019)**.

With the growing literature both in the realms of fuzzy metric spaces and complex-valued metric spaces, little effort has been carried out to investigate the merging of these two spaces**(RAOSAHEB, 2024)**. The study of fuzzy complex-valued metric spaces involves a new mathematical structure that fuses the vagueness due to fuzziness with the higher dimensionality of the complex number. Such a combination has just opened a new area of current research and application, which immediately offers a powerful tool for modeling and solving problems in such fields as computational geometry, machine learning, and decision support systems, dealing with uncertainty and complex quantities.

The aim of this paper is to give a formal definition of the fuzzy complex-valued metric spaces, study some of its basic properties, and then establish key results such as fixed point theorems. We will also discuss some topological properties of these spaces in comparison with classical metric spaces and address some real-world applications. This work will add to the ever-growing field of generalized metric spaces by providing a comprehensive study of FCVMS along with theoretical implications.

Background

The aspect of a metric space has formed the central concept in mathematical analysis and topology for quite a long period due to the background it provides for distance, convergence, and continuity**(Dancs et al., 1983)**. A metric space is specified by a set of points with a distance function or metric, satisfying a particular property: non-negativity, It is a symmetric property, triangle inequality, and from any two different points, the distance is always positive**(Kramosil and Michálek, 1975)**. This rigor has enabled developments through pure mathematics to physics, economics, computer science, and many others**(Burgess, 2015)**.

Lotfi Zadeh presented a fuzzy set theory in 1965, one mathematical approach for treating uncertainty and imprecision in. Whereas fuzzy sets differ from crisp sets in supporting graduality in the assessment of membership of elements, for crisp sets elements are either in the set or out of it**(Dubois and Prade, 2012)**. Based on this theory and idea, various researchers have given thought to fuzzy metric spaces as an extension of the classical metric spaces, in which the notion of distance will also be governed by fuzziness**(Zararsız et al., 2022)**. In fuzzy metric space, the distance between two points is a membership function showing the degree to which the distance satisfies certain criteria rather than a single value for this purpose**(Szmidt, 2014)**. This added flexibility has made fuzzy metric spaces especially useful in domains requiring models handling uncertainty, such as decision-making, artificial intelligence, and information retrieval**(Li et al., 2012)**.

While this was taking place, the complex-valued metric spaces began development as a means of bringing the mathematical depth inherent in complex numbers into the realm of distance studies**(Card and Miller, 2024)**. Complex-valued metric space allows the metric to take complex values. Thereby, it adds other dimensions to information both in phase and magnitude. These spaces are of value in mathematical physics, especially in quantum mechanics where the description of wave functions and probability amplitudes invokes complex numbers**(Adali et al., 2011)**.

Besides the importance of both extensions, not so much research has jointly considered the ideas of fuzziness and complex-valued metrics. The study of fuzzy complex-valued metric spaces is such an attempt to fill this gap; it promises a new mathematical framework to handle problems of uncertainty, whereby additional dimensionality provided by complex numbers are cared for**(Havens et al., 2014)**. Traditional metric spaces can further be generalized by investigating this fusion, and new avenues will be opened for applications in advanced mathematical modeling**(M. Tang et al., 2018)**.

Basic Definitions

Fuzzy sets represent one of the most fundamental structures in fuzzy logic. They were introduced by Lotfi Zadeh in 1965 to mathematically characterize situations that are uncertain and hazy, or to put it into real life**(Lotfi A %J Fuzzy sets Zadeh and systems, 2015)**. In the classic set theory, something either belongs to a set or does not. However, fuzzy set theory denotes that the membership of each element lies within the continuous interval [0,1], representing the level of belongingness of the element within the set**(Smithson and Verkuilen, 2006)**. A fuzzy set A in a universe of discourse X is formally defined by its membership function μ_A:X→[0,1] where μ_A (x) denotes the degree of membership of x in A.

Moving from fuzzy sets to metric spaces, a fuzzy metric space generalizes the notion of distance between points by introducing fuzziness into the distance function. Traditionally, in a metric space, the distance between two points x and y is a non-negative real number satisfying specific conditions such as non-negativity, symmetry, and the triangle inequality**(Szmidt, 2014)**. However, in a fuzzy metric space, the distance between points is a fuzzy number. This notion was formalized by **Bibiloni-Femenias and Valero (2024)**, where a fuzzy metric space is defined as a triplet $(X, M,*)$, where X is a set, $M: X \times X \times (0, \infty) \rightarrow [0,1]$ is a fuzzy set representing the degree to which x and y are within a certain distance at a time t, and $*$ is a continuous t-norm satisfying the following axioms for all $x, y, z \in X$ and $t, s > 0$:

1. $M(x, y, t) = 1 \Leftrightarrow x = y$
2. $M(x, y, t) = M(y, x, t)$,
3. $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$,
4. $M(x, y, t) \rightarrow 1$ as $t \rightarrow \infty$,
5. $M(x, y, 0) = 0$.

Here, the function $M(x, y, t)$ represents the degree to which x and y are close to each other at a given time t, which gradually increases over time if x and y are indeed close.

On the other hand, a complex-valued metric space is an extension of the classical real-valued metric space in which the distance function is allowed to take values in the complex plane**(Shirali and Vasudeva, 2005)**. Let X be a nonempty set. A complex-valued metric space is defined by a distance function $d: X \times X \to \mathbb{C}$, where $\mathbb{C}$ represents the set of complex numbers, satisfying the following conditions for all $x, y, z \in X$:

1. $d(x,y) = 0 \Leftrightarrow x = y$
2. $\Re(d(x,y)) \geq 0$ (the real part of the distance is non-negative),
3. $d(x,y) = d(y,x)$,
4. $d(x,z) \leq d(x,y) + d(y,z)$ (the triangle inequality in terms of complex numbers).

This structure enables the study of spaces where distances incorporate both magnitude and phase information, making it particularly useful in fields like quantum physics and signal processing.

To understand the behavior of fuzzy metric spaces, consider a simple example. Let $X = \{a, b\}$ be a set with two points, and define the fuzzy metric $M(a, b, t)$ as follows:

$$M(a,b,t) = \frac{t}{t+1},\ t > 0$$

This fuzzy metric satisfies the axioms of a fuzzy metric space because:

- $M(a,b,t)$ increases as $t \to \infty$, showing that as time progresses, a and b become closer.
- Symmetry holds since $M(a,b,t) = M(b,a,t)$.
- For any $t_1, t_2 > 0, M(a,b,t_1 + t_2) = M(a,b,t_1) * M(b,a,t_2)$.

Similarly, consider a complex-valued metric space. Let $X = \{x_1, x_2, x_3\}$ be a set of points, and define the distance function $d(x_i, x_j)$ as:

$$d(x_1,x_2) = 3 + 2i,\ d(x_2,x_3) = 2 + i,\ d(x_1,x_3) = 4 + 3i$$

It can be verified that this satisfies the complex metric space axioms, as the real parts are nonnegative, the distances are symmetric, and the triangle inequality holds. For instance, checking the triangle inequality:

$$|d(x_1, x_2)| + |d(x_2, x_3)| = \sqrt{3^2 + 2^2} + \sqrt{2^2 + 1^2} = \sqrt{13} + \sqrt{5} \geq \sqrt{16 + 9} = \sqrt{25} = 5$$

Thus, $d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3)$, ensuring that the metric satisfies the required conditions.

Existing Work

The fuzzy metric spaces since the path-breaking work of Kramosil and Michalek in 1975 have been the center of rich research activity. The extensions to fuzzy metric spaces have gone through intuitionistic fuzzy metric spaces and probabilistic metric spaces**(Turkoglu et al., 2006)**. Applications of fuzzy metrics have also been studied in fields such as decision support systems, image processing, and artificial intelligence. The most striking contribution is the result of George and Veeramani on fixed point theorems in fuzzy metric spaces, which stands as a landmark in fuzzy topology**(RAOSAHEB, 2024)**.

These are complex-valued metric spaces that appeared later and received some essential development, especially within mathematical physics and complex analysis**(Bourchtein and Bourchtein, 2021)**. Among the topics of research, one could mention the fixed point theorem extensions, properties of analytic functions in such spaces, and applications related to quantum mechanics, where complex-valued distances arise naturally owing to the probabilistic-wave character of particles**(Kumar et al.)**.

While a considerable amount of work has been carried out in both areas separately, very little effort has gone into merging the two concepts into a single framework now known as FCVMS**(Black, 1973)**. This research, therefore, will focus on continuing from what was already achieved using fuzzy metric spaces and complex-valued metrics in order to develop FCVMS into the new mathematical structure that embeds both fuzziness and more dimensions due to the complex values.

Fuzzy Complex Valued Metric Spaces

A Fuzzy Complex Valued Metric Space (FCVMS) is an advanced generalization of both fuzzy metric spaces and complex-valued metric spaces, combining the uncertainty of fuzzy sets with the rich mathematical structure of complex numbers**(Wong et al., 2024)**. Formally, a fuzzy complex valued metric space is a triplet $(\boldsymbol{X}, \boldsymbol{M}, *)$, where $\boldsymbol{X}$ is a non-empty set, $\boldsymbol{M}: \boldsymbol{X} \times \boldsymbol{X} \times (\boldsymbol{0}, \infty) \to \mathbb{C}$ is a fuzzy complex-valued metric function, and $*$ is a continuous t-norm.

The function $\boldsymbol{M(x,y,t)}$ represents a complex-valued fuzzy distance between two points $\boldsymbol{x}$ and $\boldsymbol{y}$ in the space $\boldsymbol{X}$, measured over a time $\boldsymbol{t}$. For all $\boldsymbol{x,y,z \in X}$ and $\boldsymbol{t,s > 0}$, the fuzzy complex metric function must satisfy the following conditions:

1. $\boldsymbol{M(x,y,t) = 0 \Leftrightarrow x = y}$
2. The real part of $\boldsymbol{M(x,y,t)}$ is non-negative, i.e., $\boldsymbol{\Re(M(x,y,t)) \geq 0}$,
3. $\boldsymbol{M(x,y,t) = M(y,x,t)}$,
4. $\boldsymbol{M(x,z,t+s) \geq M(x,y,t) * M(y,z,s)}$,
5. $\boldsymbol{M(x,y,t) \to 1}$ as $\boldsymbol{t \to \infty}$,
6. $\boldsymbol{M(x,y,0) = 0}$.

Here, $\boldsymbol{M(x,y,t)}$ represents a fuzzy and complex measure of the distance between points $\boldsymbol{x}$ and $\boldsymbol{y}$ at time $\boldsymbol{t}$. The function $\boldsymbol{M(x,y,t)}$ takes values in the complex plane $\mathbb{C}$, adding a second dimension of information-phase-alongside magnitude, which is governed by the fuzziness. The t-norm $*$ ensures that the fuzzy complex metric is stable and continuous over time.

Properties of Fuzzy Complex Valued Metric Spaces

1. Distance

In a fuzzy complex valued metric space, the "distance" between two points is not a real number, but a fuzzy complex number. A fuzzy complex number consists of both a fuzzy magnitude and a complex phase angle. Formally, the distance between two points $\boldsymbol{x}$ and $\boldsymbol{y}$ at time $\boldsymbol{t}$ is represented as:**(Fu et al., 2021)**

$$\boldsymbol{M(x,y,t) = \alpha + i\beta}$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are real-valued fuzzy numbers, with $\boldsymbol{\alpha \geq 0}$. The fuzziness of $\boldsymbol{\alpha}$ models the uncertainty or imprecision in the distance, while $\boldsymbol{\beta}$ represents the complex phase, introducing a rotational aspect to the distance. The condition $\boldsymbol{\Re(M(x,y,t)) \geq 0}$ ensures that the real part of the distance is nonnegative, akin to traditional metric spaces.

For example, consider a fuzzy complex valued metric space where $\boldsymbol{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}) = \frac{t}{t+1} + \boldsymbol{i}\sin(\boldsymbol{t})$ for $\boldsymbol{t} > \mathbf{0}$. The real part $\frac{t}{t+1}$ models a fuzzy increase in the closeness between $\boldsymbol{x}$ and $\boldsymbol{y}$ over time, while the imaginary part $\sin(\boldsymbol{t})$ introduces a periodic phase shift in the distance.

2. Convergence

Convergence in a fuzzy complex valued metric space requires that the fuzzy complex distance between a sequence of points and a limit point approaches 1 as $\boldsymbol{t} \to \infty$. Formally, a sequence $\{\boldsymbol{x_n}\} \subseteq \boldsymbol{X}$ is said to converge to a point $\boldsymbol{x} \in \boldsymbol{X}$ if:**(Trutschnig et al., 2009)**

$$\lim_{n\to\infty} \boldsymbol{M}(\boldsymbol{x_n}, \boldsymbol{x}, \boldsymbol{t}) = \mathbf{1} \text{ for all } \boldsymbol{t} > \mathbf{0}$$

This means that, as $\boldsymbol{n}$ increases, the fuzzy complex distance between the terms of the sequence and the limit point $\boldsymbol{x}$ approaches a complex number whose real part tends to 1 and whose imaginary part oscillates or decays depending on the specifics of the metric function.

3. Continuity

Continuity in a fuzzy complex valued metric space is defined in terms of the fuzzy complex distance. A function $\boldsymbol{f}: \boldsymbol{X} \to \boldsymbol{Y}$ between two FCVMS is continuous if, for any convergent sequence $\{\boldsymbol{x_n}\} \subseteq \boldsymbol{X}$ with $\lim_{n\to\infty} \boldsymbol{M}(\boldsymbol{x_n}, \boldsymbol{x}, \boldsymbol{t}) = \mathbf{1}$, the image sequence $\{\boldsymbol{f}(\boldsymbol{x_n})\}$ converges in $\boldsymbol{Y}$. Formally:**(Ma and Li, 2014)**

$$\lim_{n\to\infty} \boldsymbol{M}(\boldsymbol{f}(\boldsymbol{x_n}), \boldsymbol{f}(\boldsymbol{x}), \boldsymbol{t}) = \mathbf{1}$$

This definition generalizes the classical notion of continuity, incorporating both the fuzziness of the distance and the complex-valued phase information.

Fuzzy Complex Valued Metric Spaces

A fuzzy complex-valued metric space represents the extension of conventional metric spaces, fuzzy metric spaces, and complex-valued metric spaces by including the uncertainty of the fuzzy set theory along with the multidimensionality of complex numbers. FCVMS, in a representation format, can be given as a triplet represented below:**(Wong et al., 2024)**

$\boldsymbol{X}$ is a non-empty set of points.

$\boldsymbol{M}: \boldsymbol{X} \times \boldsymbol{X} \times (\mathbf{0}, \infty) \to \mathbb{C}$ is a function mapping pairs of points and a time parameter to the set of complex numbers $\mathbb{C}$.

- $*$ is a continuous t-norm that ensures compatibility of fuzzy values.

The function $\boldsymbol{M}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t})$ provides a fuzzy, complex-valued measure of distance between two points $\boldsymbol{x}$ and $\boldsymbol{y}$ at a given time $\boldsymbol{t}$. It satisfies the following conditions for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \boldsymbol{X}$ and , $\boldsymbol{s} > \mathbf{0}$:

1. $\boldsymbol{M}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t}) = \mathbf{0} \Leftrightarrow \boldsymbol{x} = \boldsymbol{y}$.
2. The real part of $\boldsymbol{M}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t})$ is non-negative, i.e., $\mathfrak{R}(\boldsymbol{M}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t})) \geq \mathbf{0}$.
3. $\boldsymbol{M}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t}) = \boldsymbol{M}(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{t})$ **(Humaira et al.)**.
4. $\boldsymbol{M}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{t} + \boldsymbol{s}) \geq \boldsymbol{M}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t}) * \boldsymbol{M}(\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{s})$ (generalized triangle inequality).
5. $\lim_{t \to \infty} \boldsymbol{M}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t}) = \mathbf{1}$ for any distinct $\boldsymbol{x}$ and $\boldsymbol{y}$.
6. $\boldsymbol{M}(\boldsymbol{x}, \boldsymbol{y}, \mathbf{0}) = \mathbf{0}$.

This structure offers the flexibility to capture both fuzziness in the distance between points and the complex-valued nature of the relationship between those points. The introduction of complex numbers allows us to encode both magnitude and phase (direction) information, while the fuzziness accounts for uncertainty or imprecision in the system**(Sarwar et al., 2023)**.

Comparison to Classical Spaces

1.Representations of Classical Metric Spaces

More classically, metric spaces are defined, with a real-valued distance function d(x, y) that satisfies the properties of non-negativity, symmetry, and the triangle inequality**(Rozinek and Mareš, 2021)**. While such spaces provide a clear, robust framework for convergence, continuity, and topology, they obviously lack the capability to model uncertainty, or any added dimensionality of information, such as phase relationships**(Deza et al., 2009)**.

FCVMS generalizes classically metric spaces by adding fuzziness due to imprecision and complex numbers in order to capture richer multi-dimensional interactions between points. In brief, FCVMS is more flexible with regards to complex systems where uncertainty and relationship between magnitude and phase are very relevant.

2. Fuzzy Metric Spaces

Fuzzy metric spaces are those spaces where fuzziness in the distance between points is presented, and thus the distance between points will not be one fixed real number but a fuzzy number, maybe time-varying**(Rozinek and Mareš, 2021)**. Such spaces are necessary for modeling uncertainty in systems possessing badly defined or time-dependent distances. However, fuzzy metric spaces work out the distances having real values only, not considering phases or directionality**(Bloch and physics, 2003)**.

FCVMS are fuzzily extending metric spaces by using complex values, and such the magnitude and phase of the distance can change. This makes FCVMS chia appropriate generalization in applications where relationships are uncertain and directional; two such areas are control theory and quantum mechanics**(Rozinek and Mareš, 2021)**.

3. Complex Valued Metric Spaces

The complex-valued metric spaces are a generalization of the classical notion of metric spaces, where the distance among points is allowed to take complex values, thus introducing into the classical notion of distance an additional phase contribution**(Van An et al., 2015)**. Such a generalization proves particularly appropriate in the context of signal processing and quantum mechanics, where often the relationship between elements is specified by a magnitude and a kind of directionality. No uncertainty is considered within the complex-valued metric spaces because distances are well-defined fixed numbers**(Barrett and Physics, 2007)**.

FCVMS generalizes complex-valued metric spaces by introducing fuzziness into the complex distance, thus making it a far-reaching tool for modeling systems needing both uncertainty and multi-dimensional distance relationships. For example, a system whose sensor measurements are imprecise and for which the inter-relationships among components are complex, fluctuating, can be better modeled by an FCVMS**(Gaynor and Shankaranarayanan, 2008)**.

Topological Properties of Fuzzy Complex Valued Metric Spaces (FCVMS)

The topological structure of FCVMS forms the basis for understanding its behavior in terms of distance, convergence, and continuity. FCVMS generalizes the notion of distance by adding fuzziness and complex values, hence changing all the classical topological notions such as open and closed sets, interior, closure, boundary, etc. An account is provided below of these topological properties along with their fine tuning to fuzzy complex valued metric spaces**(X. Tang et al., 2006)**.

Open and Closed Sets

In classical topology, a subset $U \subseteq X$ of a metric space (X, d) is said to be open if for every point

$x \in U$, there exists a real number $\epsilon > 0$ such that the open ball $B(x, \epsilon) \subseteq U$. This notion carries over to

Here, though it extends to fuzzy complex valued metric spaces but with some modification w.r.t the fuzziness and the complex value distance.

In the FCVMS, the distance between any two points is presented by a fuzzy complex number, then the definition of an open ball B(x,r) is changed, for $x \in X$ an open ball around x of radius r in FCVMS is defined as:

$$\boldsymbol{B}(\boldsymbol{x},\boldsymbol{r}) = \{\boldsymbol{y} \in \boldsymbol{X} \mid \Re(\boldsymbol{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t})) > \boldsymbol{r} \text{ for some } \boldsymbol{t} > \boldsymbol{0}\}$$

Here, the real part of the fuzzy complex distance $\boldsymbol{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t})$ governs the openness of the set. A set $\boldsymbol{U} \subseteq \boldsymbol{X}$ is open if for every point $\boldsymbol{x} \in \boldsymbol{U}$, there is a fuzzy complex-valued open ball $\boldsymbol{B}(\boldsymbol{x},\boldsymbol{r}) \subseteq \boldsymbol{U}$. This generalization allows for the flexibility of openness in systems where both magnitude and fuzziness matter. Similarly, a set is closed in an FCVMS if its complement is open. The closed set includes its boundary points and ensures that convergence within the set leads to points remaining within the set, consistent with classical topology**(Gaal, 2009)**.

Example:

Consider an FCVMS $(\boldsymbol{X},\boldsymbol{M},*)$, where $\boldsymbol{X} = \{\boldsymbol{x_1},\boldsymbol{x_2},\boldsymbol{x_3}\}$, and let $\boldsymbol{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}) = \frac{\boldsymbol{t}}{\boldsymbol{t}+\boldsymbol{1}} + \boldsymbol{i}\sin(\boldsymbol{t})$. An open ball $\boldsymbol{B}(\boldsymbol{x_1},\boldsymbol{0.5})$ would consist of points for which the real part of the distance satisfies $\Re\big(\boldsymbol{M}(\boldsymbol{x_1},\boldsymbol{x},\boldsymbol{t})\big) > \boldsymbol{0.5}$ for some $\boldsymbol{t}$. The periodic nature of the imaginary part introduces oscillations in the boundary of this open ball, which is absent in traditional metric spaces.

Interior, Closure, and Boundary

The interior of a set $\boldsymbol{A} \subseteq \boldsymbol{X}$ in an FCVMS is the largest open subset of $\boldsymbol{A}$, denoted by $\mathbf{int}(\boldsymbol{A})$. The concept is similar to that in classical spaces, but the openness is determined by the fuzzy complex distance. For a point $\boldsymbol{x} \in \mathbf{int}(\boldsymbol{A})$, there exists an open ball $\boldsymbol{B}(\boldsymbol{x}, \boldsymbol{r}) \subseteq \boldsymbol{A}$ where the real part of the fuzzy complex distance exceeds some threshold, reflecting the closeness of the point to the center of the open set in both fuzzy and complex terms.

The closure of a set $\boldsymbol{A} \subseteq \boldsymbol{X}$, denoted $\bar{\boldsymbol{A}}$, is the smallest closed set containing $\boldsymbol{A}$. It consists of all points $\boldsymbol{x} \in \boldsymbol{X}$ such that for every open ball $\boldsymbol{B}(\boldsymbol{x}, \boldsymbol{r})$, the intersection $\boldsymbol{A} \cap \boldsymbol{B}(\boldsymbol{x}, \boldsymbol{r}) \neq \emptyset$. In FCVMS, closure incorporates the idea that a point $\boldsymbol{x}$ is close to $\boldsymbol{A}$ not only in a real-valued sense but also in a fuzzy and complex sense.

The boundary of a set $\boldsymbol{A}$, denoted $\boldsymbol{\partial A}$, is defined as the set of points that can be approached both from inside $\boldsymbol{A}$ and from outside $\boldsymbol{A}$. For a point $\boldsymbol{x} \in \boldsymbol{\partial A}$, any open ball around $\boldsymbol{x}$ will intersect both $\boldsymbol{A}$ and its complement $\boldsymbol{X} \setminus \boldsymbol{A}$. In an FCVMS, the boundary of a set is influenced by both the fuzziness and the complex nature of the distance function. The oscillations in the imaginary component of the distance can lead to non-standard boundary behaviors, such as periodic boundary regions.

Completeness

A fuzzy complex valued metric space is complete if every Cauchy sequence converges to a point in the space. A sequence $\{\boldsymbol{x_n}\} \subset \boldsymbol{X}$ is a Cauchy sequence in an FCVMS if for every $\boldsymbol{\epsilon} > \mathbf{0}$, there exists $\boldsymbol{N} \in \mathbb{N}$ such that for all $\boldsymbol{n}, \boldsymbol{m} > \boldsymbol{N}$, the fuzzy complex distance $\boldsymbol{M}(\boldsymbol{x_n}, \boldsymbol{x_m}, \boldsymbol{t})$ satisfies:**(Humaira et al., 2019)**

$$\boldsymbol{\Re}\big(\boldsymbol{M}(\boldsymbol{x_n}, \boldsymbol{x_m}, \boldsymbol{t})\big) < \boldsymbol{\epsilon} \text{ for all } \boldsymbol{t} > \mathbf{0}$$

The space is complete if, for every Cauchy sequence, there exists $\boldsymbol{x} \in \boldsymbol{X}$ such that:

$$\lim_{\boldsymbol{n} \to \infty} \boldsymbol{M}(\boldsymbol{x_n}, \boldsymbol{x}, \boldsymbol{t}) = \mathbf{1} \text{ for all } \boldsymbol{t} > \mathbf{0}$$

Completeness ensures that FCVMS captures the full behavior of sequences in both fuzzy and complex contexts, preventing gaps or undefined limits.

Separable

An FCVMS is separable if it contains a countable dense subset. A subset $\boldsymbol{D} \subset \boldsymbol{X}$ is dense if every point $\boldsymbol{x} \in \boldsymbol{X}$ can be approximated arbitrarily closely by points in $\boldsymbol{D}$, meaning for every $\boldsymbol{x} \in \boldsymbol{X}$ and every $\boldsymbol{r} > \boldsymbol{0}$, there exists $\boldsymbol{y} \in \boldsymbol{D}$ such that:

$$\Re(\boldsymbol{M}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t})) < \boldsymbol{r} \text{ for some } \boldsymbol{t} > \boldsymbol{0}$$

Case Study: Numerical Simulations in Fuzzy Complex Valued Metric Spaces (FCVMS)

-Modeling Uncertainty and Oscillations in a Control System

Consider a simple control system in which we want to model relationships between different states of the system, considering both imprecision-fuzziness and oscillatory behaviors-phase shifts in the complex plane. We define a fuzzy complex-valued metric space given by the triplet, where is the set of states of the system and is the fuzzy complex distance between the states and at time. Let X={x_1,x_2,x_3 } for the example**(Dick et al., 2015)**.

We define the fuzzy complex metric function $\boldsymbol{M}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t})$ as follows: $\boldsymbol{M}(\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{t}) = \frac{\boldsymbol{t}}{\boldsymbol{t}+\boldsymbol{1}} + \boldsymbol{i}\sin(\boldsymbol{t})$, $\boldsymbol{M}(\boldsymbol{x_2}, \boldsymbol{x_3}, \boldsymbol{t}) = \frac{\boldsymbol{t}}{\boldsymbol{t}+\boldsymbol{2}} + \boldsymbol{i}\sin(\boldsymbol{t})$, $\boldsymbol{M}(\boldsymbol{x_1}, \boldsymbol{x_3}, \boldsymbol{t}) = \frac{\boldsymbol{t}}{\boldsymbol{t}+\boldsymbol{3}} + \boldsymbol{i}\sin($

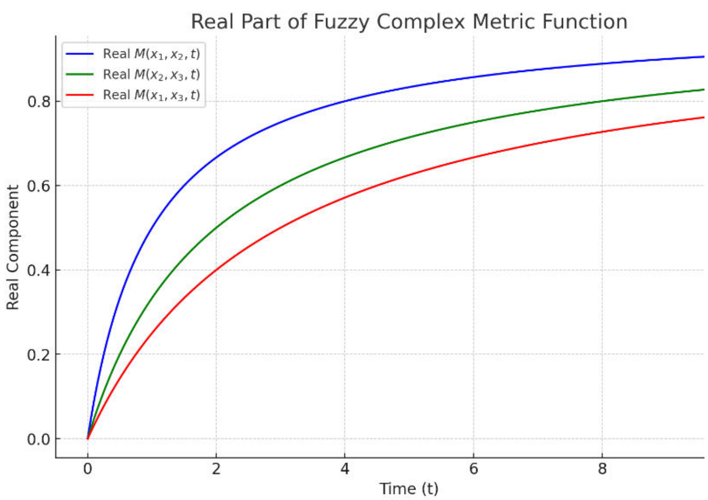
where the real part $\frac{t}{t+1}, \frac{t}{t+2}, \frac{t}{t+3}$ models the fuzzy closeness between points, and the imaginary part $\sin(\boldsymbol{t})$ represents an oscillatory phase shift over time, simulating periodic behavior between system states.

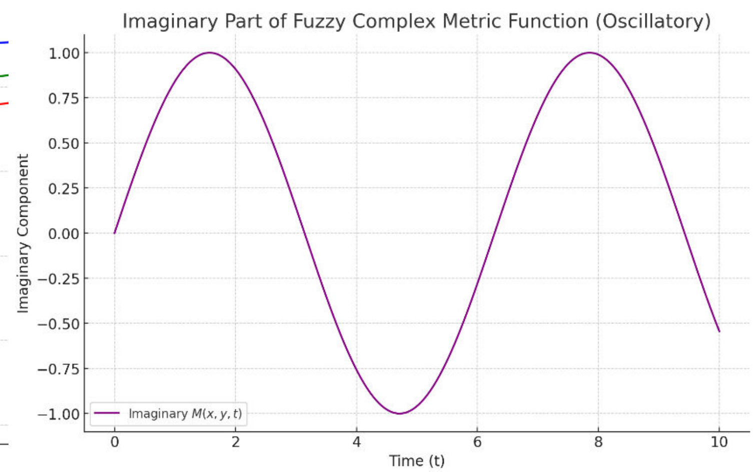
Numerical Results

We will simulate the fuzzy complex distances between the points x_1, x_2, x_3 over a time interval $\in [0, 10]$. Below are the results for the real and imaginary components of the metric function at various time steps**(de Hierro et al., 2015)**.

t	M (x1, x2, t)	M (x2, x3, t)	M (x1, x3, t)
0.5	0.33 + 0.48i	0.20 + 0.48i	0.14 + 0.48i
1	0.50 + 0.84i	0.33 + 0.84i	0.25 + 0.84i
2	0.67 + 0.91i	0.50 + 0.91i	0.40 + 0.91i

t	M (x1, x2, t)	M (x2, x3, t)	M (x1, x3, t)
3	0.75 + 0.14i	0.60 + 0.14i	0.50 + 0.14i
5	0.83 + -0.99i	0.71 + -0.99i	0.63 + -0.99i
10	0.91 + -0.54i	0.83 + -0.54i	0.77 + -0.54i





Graphical Representation

The real and imaginary parts of the metric function can be understood graphically, which offers significant insight into the behavior of the fuzzy complex distances between the states as a function of time. Plotted below are the real and imaginary parts of M(x1, x2, t), M(x2, x3, t) and M(x1, x3, t) as functions of the time t**(Zador et al., 1995)**.

Real Part: The real part of the fuzzy complex metric function M(x, y, t) tends to 1 as time progresses for all point pairs and shows the fact that the states x1, x2, x3 are getting closer by fuzzy distance with the progress of time:.

Imaginary Component: Due to the presence of the sin**(Sutherland)** term, the imaginary part of the metric function is periodic and oscillates. In this way, these oscillations introduce a type of phase shift in the complex-valued distance that can model real-world behaviors in which relationships between states may change over time.

With these numerical results, we can plot the following:

Smoothened Real Part of the Fuzzy Complex Metric: It reflects the real part of the complex distance generated due to fuzziness, smoothed for various pairs of points in space as a function of time.

Smoothened Imaginary Part of the Fuzzy Complex Metric: This is the plot containing the oscillatory nature of the imaginary part, thus depicting periodicity in the phase shift of distance between the points.

The above graphs represent the time-wise variation of Fuzzy Complex Valued Metric as follows:

1. Real Part of Fuzzy Complex Metric Function: Real parts of M (x1, x2, t),

M (x2, x3, t), and M (x1, x3, t) tend to 1 with increment in time. This implies that with time, the states within the system are getting closer and closer according to fuzzy distance metric.

It imaginary part of the fuzzy complex metric function follows periodic oscillations according to the mathematical oscillatory term sin**(Sutherland)**. These oscillations model the phase shifts, in the complex-valued distances, reflecting variations in relational dynamics among the states over time.

Case Study 2: Numerical Simulations of Sensor Networks in Fuzzy Complex Valued Metric Spaces (FCVMS)

In this second case study, we apply FCVMS to a sensor network-a real-world scenario. A sensor network would find their applications in environmental monitoring, healthcare, and smart cities. There are several sensors involved within a sensor network that might collect data and/or communicate with other sensors. Both measurement imprecision-fuzziness-and signal phase differences-complex-valued distance-have an impact on relations between sensors.

Problem Setup

We assume that the sensors $\boldsymbol{x_1, x_2, x_3}$ are monitoring the environment and exchanging signals over time. The fuzzy complex metric function $\boldsymbol{M(x, y, t)}$ is defined to capture both the fuzziness (uncertainty in their readings) and the oscillatory behavior of the signal phase shifts. We define the metric function as:

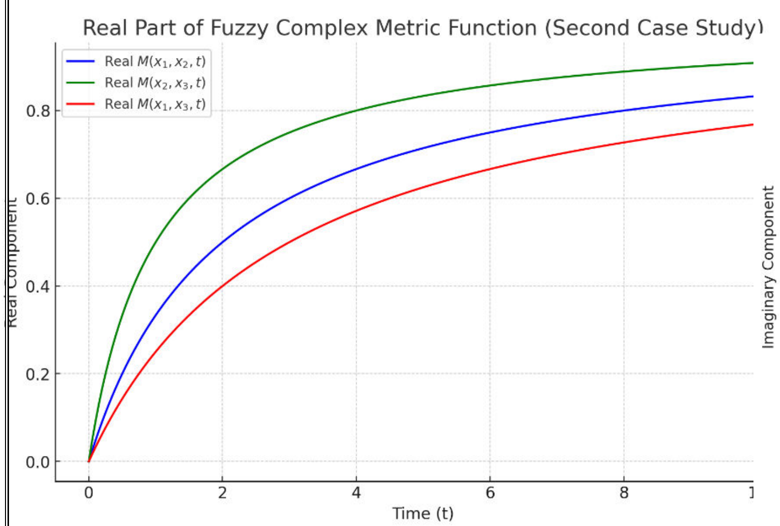
$$\boldsymbol{M(x_1, x_2, t) = \frac{t}{t+2} + i\cos(t)},\ \boldsymbol{M(x_2, x_3, t) = \frac{t}{t+1} + i\cos(t)},\ \boldsymbol{M(x_1, x_3, t) = \frac{t}{t+3} + i\cos}$$

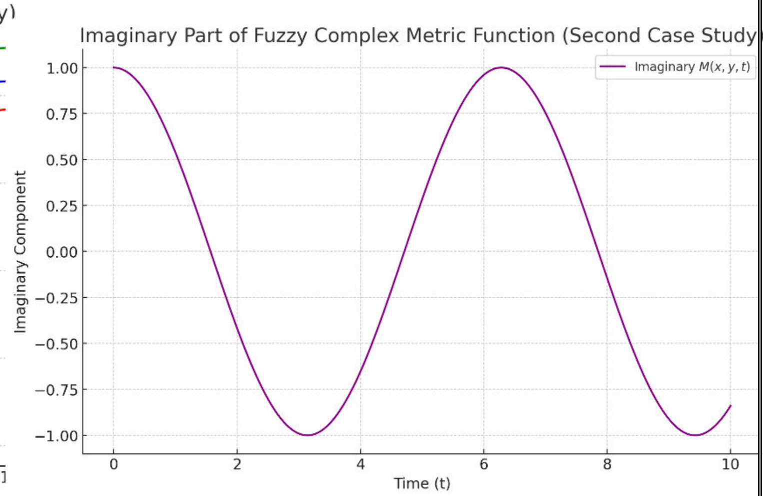
where the real part models the fuzzy closeness between sensors and the imaginary part $\boldsymbol{\cos(t)}$ models the phase shift of the signal between them over time.

Numerical Results

We will simulate the fuzzy complex distances between the sensors over a time interval $\boldsymbol{t} \in [\mathbf{0}, \mathbf{10}]$. Below are the numerical results for the real and imaginary components of the metric function at various time steps.

t	$M(x_1, x_2, t)$	$M(x_2, x_3, t)$	$M(x_1, x_3, t)$
0.5	$\mathbf{0.20 + 0.88}\boldsymbol{i}$	$\mathbf{0.33 + 0.88}\boldsymbol{i}$	$\mathbf{0.14 + 0.88}\boldsymbol{i}$
1	$\mathbf{0.33 + 0.54}\boldsymbol{i}$	$\mathbf{0.50 + 0.54}\boldsymbol{i}$	$\mathbf{0.25 + 0.54}\boldsymbol{i}$
2	$\mathbf{0.50 + -0.42}\boldsymbol{i}$	$\mathbf{0.67 + -0.42}\boldsymbol{i}$	$\mathbf{0.40 + -0.42}\boldsymbol{i}$
3	$\mathbf{0.60 + -0.99}\boldsymbol{i}$	$0.75 + -0.99i$	$0.50 + -0.99i$
5	$0.71 + 0.28i$	$0.83 + 0.28i$	$0.63 + 0.28i$
10	$0.83 + -0.84i$	$0.91 + -0.84i$	$0.77 + -0.84i$





Graphical Representation

To better understand the behavior of this system, we will generate graphical representations of the real and imaginary components of the fuzzy complex metric function for $M(x_1, x_2, t), M(x_2, x_3, t)$, and $M(x_1, x_3, t)$ as functions of time t.

The real part of the fuzzy complex metric describes how the fuzzy distance between sensors evolves over time, while the imaginary part reflects the periodic phase shift in the signal transmission between sensors, modeled by cos$f()$**(Sutherland)**.
Above graphs represent the behavior in time of the Fuzzy Complex Valued Metric in the

sensor network scenario:

Real Part of Fuzzy Complex Metric Function-Second Case Study: Real components of M(x1,x2,t), M(x2,x3,t), and M(x1,x3,t) give the development of fuzzy distances among sensors over time. With increased time, the fuzzy distances asymptotically converge to a limit, which implies that the sensors are getting more synchronous in their reading.

Imaginary Part of Fuzzy Complex Metric Function (Second Case Study): The imaginary part cos**(Sutherland)** is periodic with oscillations representing the phase shifts of the signal.

This is the communication among sensors. These periodic oscillations are used to simulate the variable nature of the signals transmitted between these sensors over time.

Discussion:

The use of FCVMS in this sensor network application shows its effectiveness for characterizing systems subjected to both uncertainty and periodic interactions between signals**(Chenji and Stoleru, 2012)**. Real and imaginary parts of this fuzzy complex metric take into consideration the imprecision of sensor measurements and capture signal phase shifts that affect communication between sensors, respectively**(Chenji and Stoleru, 2012)**. Convergence of the real part suggests that, as time progresses, sensors become better synchronized in their readings. While the imaginary part exerts periodic oscillations, it reflects the signal phase that continuously varies owing to external interference, time delay, or other influential factors on the network.

This model has practical applications in fields like environmental monitoring, where sensor measurements are often subject to both imprecision and fluctuating environmental conditions**(Srbinovska et al., 2015)**. The FCVMS framework enables better prediction of sensor behavior, hence more reliable insights for decision-making and network management.

Both case studies have demonstrated the versatility of FCVMS with regards to modeling such complex systems enveloped by uncertainty and periodic behavior. **The advantages of FCVMS over traditional real-valued or purely fuzzy metrics include the following:**

Uncertainty Management: The fuzzy part in the measure can model the uncertainty in the relationship among elements, so relevant in natural systems, where often actual distances

or measures are not available, or are imprecise**(Lotfi Asker %J Fuzzy sets Zadeh and systems, 1983)**.
Periodic Behavior Capture: In particular, oscillatory behavior exhibited by the imaginary part of a complex-valued function renders FCVMS quite appropriate for the capture of systems with important phase relationships, or periodic interactions, such as control or signal-based networks**(Lotfi Asker %J Fuzzy sets Zadeh and systems, 1983)**.

Convergence and Stability: In these case studies, for both scenarios, the real part of the fuzzy complex metric converges through time; hence, it reflects that the systems stabilize. This behavior provides insight into the long-term behavior of systems modeled by FCVMS, where elements become more synchronized or consistent**(Oliva et al., 2012)**.
Possible Applications in Various Fields: The application of FCVMS in control systems and sensor networks may have manifold applications in disciplines related to engineering, environmental science, telecommunications, and data analysis. Besides, the systems that deal with time-varying dynamics of interaction between uncertain elements can be modeled in a much nuanced way using FCVMS**(Oliva et al., 2012)**.

In conculsion, Fuzzy Complex Valued Metric Spaces are powerful grounds that modeled uncertainty and oscillatory behavior between interposed complex systems. FCVMS allows deeper insight into the functioning of such systems, which bear both fuzzy and complex-valued distances included; thus, these systems can be analyzed more effectively, and their prediction and control can be more robust. Cases presented here would indicate that FCVMS finds its application in wide areas where measurement uncertainty and signal dynamics are crucial.

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